TRANSVERSAL NON-SUBSTITUTION THEOREMS

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ABSTRACT
The mathematical transversality in this paper yields Economically transversal non-substitution theorems for technology and market development level in which any non-transversal technology vectors remain unpriced. This result implies imports and exports to be complements, not substitutes. These theorems are reached by giving a uniquely path-lifting universal covering space for a Riesz kernel-like property for unique price functions to construct a global regular-valued equilibrium manifold. The universal covering formulation is justified through invariance of domain and dimension, and Borsuk-Ulam in the argument of a trivial fundamental group that enables a Sard-like transversality. A theorem is also given for a transversal measure space for the equilibrium proof using Dirac measure.

JEL Codes: C6; D5
Keywords: general equilibrium; diffeomorphisms; smooth manifolds; Sard's theorem; transversal fixed-points; fundamental trivial group; Borsuk-Ulam; Invariance of Domain; transversality; non-substitution theorems

INTRODUCTION

Differential topology has been a useful tool for Economic analysis for general equilibrium as used by [14; 8; 15]. Balasko in [14] uses the argument of covering as ramified covering as a finite covering of regular values in the sense of Sard's theorem. This paper extends this notion of covering maps into the universal covering argument through uniting the topological properties of dimensional regularity (implied in invariance of dimension and domain, Brouwer's fixed-point theorem, no-retraction principle [1; 4; 5; 6; 7] and Sard-Brown theorem) with the concept of trivial fundamental group in topology from [2]. The unique path-lifting property implies an Economic meaning similar to Riesz Representation Theorem in Financial Economics and through a mathematical meaning of non-uniqueness of critical or singular values in the smooth manifold [12; 13; 14]. This mathematical formulation is given for an application of this analysis in this paper that is related to a closure under measure zero, as in [14], of the set of critical values on the boundary of the smooth and convex manifold, as in [13], such that a truly new technology vector, t+ is an unpriced critical value with measure zero. This implies that in the regular-valued manifold (in the sense of Sard) the priced economies taken as regular submanifolds, for instance, of priced technology vectors t and consumption or market development level vectors m are mutually non-substitutable or complementary. This simultaneously implies the imports and exports of an economy to be complements too; as is empirically confirmed by [18]. This means that in the regular-valued manifold a given regular submanifold cannot take the place of another manifold in magnitude or coordinate uniqueness terms. This simply means the translation of mathematical transversality, like given by [8], into the Economic transversality condition, according to which, the optimality of the rate of resource utilization demands that as time→∞, unused economic value v→0 [16]. The Theorem 2.1 gives a new convexity proof of atomless measures for a transversal measure space through a joint formulation of Sard's theorem by the application of no-retraction principle in terms of Dirac measure.

Covering spaces is a compactness result which can be related to fundamental groups and invariance of domain in topology. The universal covering is the expression of this element [2]. The invariance of domain and Brouwer's fixed-point theorem through topological invariance of dimension, as in this paper, connect the universal covering and fundamental trivial group concepts to differential topology in the form of Sard-Brown [13] and the proof of Brouwer's fixed-point and no-retraction (also no smooth retraction) theorems. Consider the following from [2]:

"If x is a generator of infinite cyclic group G and y belongs in an arbitrary group H then there is a unique homomorphism h : G→H such that h(x) = y and $h(x^n) = y^n$ for all n. The fundamental group of circle is infinite cyclic which implies no continuous retraction from a trivial $()^2$ (convex) to a non-trivial $()^1$ (like circle's) fundamental group." [2]

Results:

1.0. Transversality and Universal Covering:

1.1. Proposition: A smooth and convex manifold $X^n \subset \mathbb{R}^n$, with submanifolds of equal dimension M, N, has a fixed-point for a smooth map f : M →N if and only if all the critical values, Δ, of non-zero Lebesgue measure lie on the boundary dX^{n-1}.

For proof see, "Corollary (A. B. Brown) . The set of regular values of a smooth map f : M → N is everywhere dense in N." And "Lemma 2" (page 12), Milnor's topology [13]. And "Lemma 5" (page 14) [13]. And "the set of singular values Σ, is closed with measure zero". [14]

1.2. Proposition: There are C^r diffeomorphisms in $X^n \subset \mathbb{R}^n$, a smooth and convex manifold, such that X^n has a unique and global fixed-point with all the critical values, Δ, of non-zero Lebesgue measure lying on the boundary dX^{n-1}.

(For proof see, from "Definition", "Lemma 3" and "Lemma 4" (pages 12-13), [13]. And "Lemma 5" (page 14), [13].)

1.3. Theorem: Let $X^n \subset \mathbb{R}^n$ be a smooth manifold with convex differentials (diffeomorphisms) and with a uniquely path-lifting universal covering space X*, if: (i) X^n has transversal submanifolds with transversal fixed points, (ii) all the critical values Δ, in the smooth map $f : X^n \to X^n$, of non-zero Lebesgue measure lie on the boundary dX^{n-1}. And that, (ii) ⇔ (i).

1.3.1. Proof: Suppose if (ii) does not hold then there are some critical points $c \in intX^n$ such that no smooth retractions $Df : C^1 : f(c) : X^n \to X^n$ exist in the $intX^n$ it implies that $intX^n$ is not convex whereas, for the Invariance of Dimension for smooth C^1 diffeomorphisms [3], it contradicts the no smooth retraction principle from the boundary dX^{n-1}.

From Sard: "If f : M → N is a diffeomorphism then dfx : TMx → TNy is an isomorphism of vector spaces. In particular dimM = dimN." (Proof from Milnor's Topology.) [10;13]

1.3.2. Universal Covering Space: "The function p : X* → X is a universal covering map and a covering space p* : X*→X is isomorphic to the universal covering with naturally a unique path lifting property if and only if X* is simply connected." [2]

1.4. Transversal Fixed Points: Theorem: If M, N ⊂ X are transverse, M ⋔ N, regular submanifolds then M∩N is also a regular submanifold, of dimension dimM + dimN − dimM, then by the invariance of dimension with no smooth retraction from the boundary it implies the existence of a fixed-point between M and N.

Proof: Let M, N ⊂ X be regular submanifolds such that every point p ∈ M ∩ N satisfies TpM + TpN =

TpX. [10] On tangent spaces from [10] is the following. Let, m : M→X and n : N→X be the inclusion maps. Then we take TpM, TpN to be the images of the derivatives of m and n, in charts for M, N and X. Transversality then requires that these images span $\mathbb{R}^n$ **:** let this be denoted as #, where n = dimX. And here through Sard's theorem we reach the global condition on X^n having no critical values of Sard's non-zero Lebesgue measure in the $intX^n$ by #. And precisely here the # implies the simply connected X*, and p*, as the universal covering space. This connects transversality with unique path-lifting property of a universal covering. Considering Covering Space and Uniquely Path-lifting property from [2] the following is given.

1.5.1. Borsuk-Ulam: If X, with universal covering space X*, has a continuous antipode preservation, then X* also has it such that f(x*) = f(-x*). Whereas, f(x) = f(-x) ⇔ M ⋔ N by #. And because X* through p* is simply connected, X* is a universal covering space of X.

Proof: Adapted From [2]: "Let p, q and r be continuous maps with p = r ∘ q, q : X* →Y, p : X* → X, and r : Y → X. And if p and q, and p and r, are covering maps then so are r and q respectively." If p : X* →X be a covering map and X* being simply connected, if there be any covering map r : Y →X, there is a covering map q : X* → Y such that r ∘ q = p which shows that X* is the universal covering of X because it covers every other covering of X. And if X* is regular, in the sense of Sard, with respect to any covering of X then any regular submanifolds M, N of X have transversally connecting coverings under X*.

1.5.1.1. Remark: X*, being the universal covering of regular-valued X, must be regular in the sense of Sard.

Proof: if X* is a universal covering by 1.5.1 then X* cannot have dimension less than that of X and it cannot have dimension anywhere less than that of its own everywhere else, to keep antipode preservation.

1.5.2. Remark: Given a linear pricing kernel, to ensure Law of One Price[1], through Riesz Representation Theorem from [12], the universal covering uniquely covering every other covering is ensured.

Implying transversality as in [8], Sard [13; 14] and invariance of dimension [3], consider the atomless measures.

2.0. Transversality for Sard-like Measures:

Given that Euclidean space is first countable (metrizable) and second countable (separable), from [17] it is justified to take a transversal Borel space conception for a transversal measure space in the following. This will combine Sard's theorem with Radon-Nikodym[2] derivative by implication for transversality and uniqueness results.

2.1. Theorem: All the critical values' set Δ, for smooth maps f : X → X, in a transversal measure space (X,Σ,μ) constitutes a set of Lebesgue measure zero as v(dX) = v(Δ) = 0, ∀ Y ∈ intX, and, all regular values π have Dirac measure μ(Y) = μ(π) = 0, ∀ Y ∈ intX. Where dX and intX are the boundary and the interior, of the convex set X, respectively.

Proof: Lebesgue measure v(Δ) = 0 implies the intX to be convex and therefore it also implies the

[1] Zero degrees of freedom for equality requirement between the number of unknowns and the number of independent equations [14].

[2] "For a σ-finite measure space (A,Σ,μ) and a σ-additive, σ-finite set function v : Σ→R* there is a unique decomposition v = v1 + v2 where v1 and v2 are σ-additive and σ-finite, such that v1 is singular with respect to μ and v2 < μ. In addition, there is a finite valued measurable f : A → R such that v2(E) = ∫E fdμ, ∀ E ∈ Σ, f is unique in that if there is a function g such that v2(E)= ∫E gdμ, ∀ E ∈ Σ, then f = g except on a set of zero measure." [9] This f is the Radon-Nikodym derivative.

atomlessness[3] of measures in the intX. Following [9], given a measure space (X,Σ,μ) for every Y ∈ Σ in the intX, Dirac measure μ(Y) = 0. If Lebesgue measure v(Δ) > 0 for a Y ∈ Σ it implies that Dirac measure μ(dX) = 1 for the intX and Δ ∈ intX, which violates the convexity and atomlessness of the intX. If v(Δ) = 0, it implies, following [9], that μ(Y) = μ(π) = 0, ∀ Y ∈ intX. Thus Δ ∈ dX.

2.2. Coordinate Neutrality Lemma: Given the differentiability property of smooth maps in terms of local coordinates, if the set of critical values Δ on the boundary dX^{n-1}, of the convex and smooth manifold X^n, is closed under X^n then the individual critical values are coordinate neutral or non-unique.

This simply implies non-uniqueness and zero measure properties of critical values. And coordinate dependence, continuity of price correspondences, Law of One Price and regular values imply Walras' law, like this uniqueness [14]. The following will generalize this non-uniqueness of critical/singular values through singular measures.

2.3. Transversal Theorems:

Now we move towards deriving the economic transversality [16] condition from the mathematical ones above.

2.3.1. Theorem: For every nonnegative final exports vector z ≥ 0 (from a transversal submanifold Z in a convex economy X ⊂ $\mathbb{R}^n$), there is a nonnegative technology vector t ≥ 0 which is, for transversality condition, completely priced in the consumption m ≥ 0, such that, any t+ vectors of critical values, where t ≠ t+, have Lebesgue measure zero for z (that is, the only critical values found are of measure zero). That is, for Z, M and T ⊂ X,

$\{ z \in Z \pitchfork m \in M \pitchfork t \in T \} \Leftrightarrow p(t+) = 0, \forall p \in P \subseteq intX, \forall t+ \in dX^{n-1}$.

Where M is the domestic consumption of the exporting economy.

Proof: The t+ critical values of technology are the boundary case of dimension n-1 in terms of Brouwer and Sard and therefore remain unpriced as p(t+) = 0. The completely priced t implies, and is transversal with, a nonnegative vector m ≥ 0, for m ∈ M, the market development level[4], such that m is non-substitutable for t. That is, given the value of technology as v(t), and Economic transversality condition demanding that an optimum use of a value v, at time < ∞, must be such that, when time→∞, v→0, we have the following: v(t)→0 as time→∞ ⇔ m→0 when time→∞. This follows from the fact that the price of the critical values of technology remains unpriced as, p(t+) = 0, as the boundary case of the dimension n-1. And that this result does not hold, as Walras' law does not hold in terms of Sard [14], if t+, the critical values' vector of technology, is not of measure zero for a time < ∞. "Every nonnegative demand vector can be exactly produced by some efficient process" [11].

2.3.2. Theorem: For every nonnegative final consumption demand vector, m ≥ 0, of a net-importing economy in terms of its imports y, there is a nonnegative and non-substitutable technology vector, t ≥ 0, of exporting economy that is transversal with the nonnegative vector, y ≥ 0, of imports of the importing economy, such that there is no non-transversal and critical value with Lebesgue measure

[3] "A measure space (A,Σ,μ) is atomless if for every E ∈ Σ with μ(E) > 0 there exists F ∈ Σ and F ⊂ E such that μ(E) > μ(F) > 0. This rules out any individual having positive "weight" or "influence". That is, if a*, an agent is "decisive" for A in that his preferences determine those of the society as a whole then we define the Dirac measure as follows: μ(E) = 1 if a* ∈ E, then ∀ E ∈ P(A), μ(E) = 0, where P(A) is the power set of A with all, including the empty, subsets of A." [9]

[4] This, as mentioned above, is best characterized as an 'effective market knowledge' which best translates into aggregate consumption or per capita real income.

> 0 between the technology t and imports y. That is, for Y, M and T $\subset$ X,

$$\{ y \in Y \pitchfork m \in M \pitchfork t \in T \} \Leftrightarrow p(t+) = 0, \forall p \in P \subseteq intX, \forall t+ \in dX^{n-1}.$$

Where M is the domestic consumption of the importing economy.

Proof: If terminal-transversality condition is defined as, ∞ = non-net-importing country, then, $y \rightarrow 0$ as time $\rightarrow \infty \Leftrightarrow$ technology $t \rightarrow 0$ as time $\rightarrow \infty$, when $m \rightarrow 0$ as time$\rightarrow \infty$. This also follows from the fact that the price of the critical values of technology remains unpriced as, p(t+) = 0, as the boundary case of the dimension n-1. And that this result also does not hold, as Walras' law does not hold in terms of Sard [14], if t+, the critical values' vector of technology, is not of measure zero for a time $< \infty$.

Discussion:

Theorems 2.3.1 and 2.3.2 give the application of all the preceding mathematical formulations. The 2.3.1 states the non-substitutability, for a vector of exports z, of regular-valued priced technology t and market development level m in the form of mathematical transversality for both of these being priced and therefore regular-valued. Such that, any t+ vector of a truly new technology has Lebesgue measure zero because of being critical-valued, in the sense of Sard, and is therefore unpriced. The mathematical transversality and regular-valued mutual non-substitutability of the t, m and z, renders the Economic transversality condition in the proof. Similarly, 2.3.2 states that the technology endowment t of an exporting country, imports y and domestic consumption m of a net-importing country, all three are transversally non-substitutable, priced and therefore regular-valued. This implies that the instance of becoming a non-net-importing country demands that the exporting country's technology endowment t which is in regular-valued relation with the importing country's consumption m, related to goods y, should come to an end. This is because m is transversal and regular-valued with imports y. Or, the technology endowment of the exporting country is acquired by the importing country itself but that is only possible if there is a market development level m in the importing country itself which would have incentivized the domestic production of the goods imported as y.[5] That is, the non-substitutability between t and m implies that there is a regular-valued non-substitutability between the imports y and exports z; implying imports and exports to be complements, not substitutes, as also empirically shown in [18]. This is rendered through Economic transversality condition in the proof too.

The import substitution (see footnote-5 for the working intuition) has an eventual chance at regular-valued development when, for instance, an exporting and technologically more competitive economy starts facing inreasing labor costs that make its exports less competitive.

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[5] But the market incentive through m, to produce the goods y, is itself predicated on the technological knowledge t of the goods y. This t again is only possible if there is a market incentive m to render t, for producing goods y, cost competitively in terms of the scale of consumption demand and therefore for the economies of scale for production. If a technology, for producing goods y, is truly and disruptively new as t+ which means there is no domestic (or foreign) m for consumption of the goods y, because of no priced technology t, then t+ remains unpriced and therefore un-rewarded in the form of too low returns and too costly or too scarce local labor-skill availability for it. Towards a new equilibrium intertemporally as t+ develops its corresponding m+ such that t+ enjoys an absence of competition and growing economies of scale it eventually gets priced as t with its regular-valued m in the new equilibrium. Yet the inter-economy/country cost competitiveness in trade is always already regular-valued such that the impulse to produce the *importable* goods y domestically is never economical or regular-valued and therefore implies a measure zero. See [19] regarding how an import-biased technical change improves the terms of trade for the home country and deteriorates them in the export-biased technical progress.

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