A THEORY OF CONVEX DIFFERENTIALS IN ECONOMICS

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Received: 11 December, 2023 **Revised:** 21 December, 2023 **Accepted:** 25 December, 2023 **Published:** 31 December, 2024

ABSTRACT

This paper proposes a concept of convex and linear functions as convex differentials among subsets of commodity and price spaces in a convex and Euclidean space linearity. These subsets are tied together in the whole commodity and price space through fixed-points' equilibrium structure in the form of nonnegative price functions under General Equilibrium Theory. The Economic outcome is a necessarily mutual pricing imperative between a new technology and its market development aspect implying imports and exports to be complements, not substitutes. The theoretical method adopted here is for a mathematical existence result.

Keywords: technological innovation; convex differentials; market development; fixed-point theorems; Farkas' lemma

INTRODUCTION

Following (Debreu, 1959) and (Starr, 2011) the fixed-point theory of mathematics is used as an established framework for General Equilibrium analysis in Economic theory. The grand intuition the subsequent theorems and their proofs in this paper will impart relates to the impossibility of an 'effective-but-truly-new' and under-developed technology with its under-developed and therefore unpriced and un-developed market. This implication about such technology with incomplete returns has a bearing on let's say an exclusive 'emphasis on exports'. The exports must depend on a technology which must be more competitive and efficient than the given imports of an economy but such exports and such a technology first need a development and pricing such that they get convexified inside the developed and priced economy with a certain amount of developed returns. This latter part is more plausibly realistic in terms of developing returns in the domestic market with resource, capital, knowledge and labor incentives in a 'localized payoff setting'. But the trick is in finding the technology and market development in a domestic economy setting to be contingent upon the competitive imports of such a domestic economy, for example, in terms of customer base, technological knowledge and concomitant financial linkages, as explained in the next section. In short, the convexification (continuity up to equilibrium prices P) and therefore the effectiveness of a new technology and its market development with their bearing upon the exports' competitiveness demand that in order for an economy to increase its exports it should increase, or particularly un-inhibit, its imports. This implication then necessitates a convex economy concept of a theoretically globalized set of economies with local and/or global 'boundary cases' (to be given below) of unpriced new technologies' development in the economic sense. Fortunately, there does exist an empirical evidence (Gomez-Sanchez, 2021) on exports and imports being complements, not substitutes, in growth-oriented industries as the present paper shows.

LITERATURE REVIEW

(Debreu, 1959) systemized the technology and commodity sets into convex priced duals in an evolution from so-called "old belief" to the mathematical proofs for the existence of equilibrium given some minimal conditions. (Starr, 2011) gives a standardized form of fixed-point theorems of Kakutani and Brouwer in the convex and priced economy sense. This paper builds on this convex economy to a *convexified* global case of continuous functions and differentials for a certain application at hand, namely, the trade balances. The continuity, as implied in (Balasko, 2009), means a small variation in commodity space must translate into small variations in equilibrium prices. This paper extends this continuity through convex differentials among the subsets of both the commodity as well as the equilibrium price spaces. Speaking in terms of *transversally* generic but extant equilibrium it must be that a *good* (with *positive price* necessarily) and its *equivalent* of let's say its price; or an *export* for an *import* equivalently; must eventually be *complementary*. (Gomez-Sanchez, 2021) is an exact empirical implication of the theoretical result of this paper in which in a stand-alone applied case the exports and imports being complements is supported by evidence.

Methodology & Results

Definition 1

A convex differential $D \in \mathbb{R}^n$, of the usual form $dy = (y'\ dx)$ as a convex linear approximation, is proposed as a Euclidean distance, a linear function, between any two points as $d(p, q)$ for all points p and $q \in \mathbb{R}^n$.

Proposition 1

As in Definition1, there exists a Euclidean distance, a continuous function, as a convex differential $D = \| d - d+ \|$, given a $d+(p+, q+)$ and $d(p, q)$ for all points p, q and p+ and $q+ \in \mathbb{R}^n$.

Lemma 1

Take two subsets $X, Y \in \mathbb{R}^n$ where X and Y are the consumption and production sets respectively then there are $x \in X$ and $y \in Y$ such that $f(x) : X \to \mathbb{R}$ and $f(y) : Y \to \mathbb{R}$.

A lift $h : X \to Y$, and a lift $g : Y \to X$, both through $\mathbb{R}^n$,

Also,

$f : X \to X \in \mathbb{R}^n$ and $f : Y \to Y \in \mathbb{R}^n$

Take a convex economy $E \in \mathbb{R}^n$ with subsets X , Y $\subset$ E. The lifts h and $g \in E$ imply a unique clearing price vector as an existence condition of equilibrium in E of the form such that there exists at least one fixed-point $f(a) = a$ for an $a \in E$. This implies convexity of E. (Brouwer's Fixed-point Theorem)

Remark 1

There can be direct fixed-point functions between sets X and $Y \subset E \in \mathbb{R}^n$. These maps can be conceived as convex linear differentials as a Euclidean distance D.

Theorem 1

A subset $M \subset E$ and the subset $T \subset E$ are proposed where M is the market development level and T as technology set; both with the lifts,

$h : X \to M$ and $g : Y \to M \in E$

And similarly for T the lifts,

$p : X \to T$ and $q : Y \to T$

h, g, p, and q can be conceived as convex differentials D between M and $T \subset E \in \mathbb{R}^n$. (The motivation for it will become clear in the following)

Statement

There are mutual fixed-points among X, Y, M and T $\subset E \in \mathbb{R}^n$

Lemma 2

Let's have n-number of open exchange economies/sectors/industries trading with each other as Ei, where i = 1…n.

Take two economies Ei and Ej with subsets Xi, Yi, Mi, and Ti and Xj, Yj, Mj and Tj respectively. Then there can be En economies with Xn, Yn, Mn, and Tn subsets ∈ En. Similarly exports set An and imports set Bn ∈ En are proposed.
Then there are n-number of functions fn among all subsets of Ei and Ej and then similarly in the subsets ⊂ En producing a unique, or a unique set of, fixed-points through the mutual convex differentials Dn among all subsets ⊂ En.

Remark 2
The inter-economies dynamics have a real-world continuity and existence in the form of the principle of Purchasing Power Parity and the Law of One Price ⇔Xn ∩ Yn ∩ Mn ∩ Tn ∩ An ∩ Bn, all ⊂ En ∩ $\mathbb{R}^n_+$ ≠ ∅.

Corollary 1
Let there be exports and imports subsets An and Bn ⊂ En respectively with their mutual convex differentials Dn. Given the subsets Xn, Yn, Mn, Tn, An and Bn ⊂ En, there is an existence of a unique equilibrium in intra-economies and inter-economies space En ∈ $\mathbb{R}^n$.

Corollary 2
An increase in, for instance, an Ai ⊂ An ⊂ En ∈ $\mathbb{R}^n$, must be implied by the lifts through Mn, Xn, Tn, Yn, Tn and Bn, all ∈ $\mathbb{R}^n$.

Note
Now it is handy to understand M as a market development level in the sense of real income or consumption level. This will become obvious after the proof below.
(Brouwer's Fixed-point Theorem and No-Retraction Theorem):

Proof 1
A convex set E ∈ $\mathbb{R}^n$ and a subset T ⊂ E (or the exports subset A ⊂ E) while C is the boundary of E. (Where C can also be taken as a vector T+ of technology on the boundary of economy E and a vector A+ of exports on the boundary of E is also an equivalent of C. Let's say T+ and A+ are vectors of technology and exports in a non-convex boundary case which means they are not integrated in the convex, developed and priced structure of the economy. The whole formulation of M+ and A+ hinges on T+.)
There does not exist a retraction r : E → C on the unit disk E ∈ $\mathbb{R}^2$ of the form r(a) = a for all a ∈ C.

Proof
Take r : E →T+ be a retraction to boundary of unit disk or unit sphere in $\mathbb{R}^n$ for producing a contradiction. Remove points a, b ∈ T+ where a, b can be taken as some technologies as some new machine with a related new labor skill both complementing each other in E. Now there are two disjoint open arcs T+\{a,b}. Let A= r^{-1}(a) and B = r^{-1}(b) and a ∈ A and b ∈ B, where A and B are some arbitrary sets. Now A and B intersect T+ but still given the r is a continuous function and{a, b}, and therefore A and B, are closed. Only at points a and b sets A and B can intersect T+ because only a and b, from A and B, are in T+. The closure of (T+\{a, b}) = T+. There is a subset of closure of (E\(A∪B) ⊃ T+ = P while the set P is open and path-connected. P is the set of price functions. A closed arc of T+ is T+a which contains a. T+a has endpoints, xa and ya, in the closure of P and the closure of P is path-connected but given E\(A∪B), the path connecting xa and ya cannot intersect A or B due to, for instance, an absence of the needed M+ (where M ⊂ E, is the market development set). P implies any set from within E, not from the boundary T+, therefore any path within E is connected. So the path xa and ya when unioned with T+\{a, b} is another connected set. Retraction image r^{-1} of the union path is T+\{a, b} because the path circumvented A and B sets which is impossible. "The image of a connected set under a continuous function cannot be disconnected". (Buxton, 2016)
As r(x)⇎r^{-1}(x) so the retraction r cannot exist.

Brouwer's Fixed-point Theorem:

Definition: A vector field on B^n in $\mathbb{R}^n$ is an ordered pair (x, v(x)) where $x \in B^n$ and v is a continuous map of B^n into $\mathbb{R}^n$. A vector field is non-vanishing if $v(x) \neq 0$ for all $x \in B^n$. (Munkres, 2016)

Theorem: "Given a nonvanishing vector field on B^2 unit disk (or alternatively a ball) there exists a point of boundary S^1, the non-retracting non-deforming unit circle, where the vector field directly points inward and then a point in S^1 where the vector field directly points outward." (Munkres, 2016)

Remark 3

It is impossible for a continuous function to not have a fixed-point. (Buxton, 2016)

Consider from (McLennan, 2018):

"Farkas Lemma: If C is a closed convex cone, then for any $b \in \mathbb{R}^n \setminus C$ there is $n \in C*$ such that $(n, b) < 0$. Where $C*$ is the polar dual of C."

This simply means that price cannot be negative[1] which implies the free disposal in Economic theory. If C here is taken as commodity space and $C*$ as the price projection dual of C then any imaginary commodity b not belonging in C cannot have a positive pricing n in $C*$ which means b is not priced and therefore not real in the sense of being a commodity.

Remark 3.1

If Farkas' lemma holds for a price space $P \subset E \in \mathbb{R}^n$, then there exists a real-valued correspondence φ(P): $P \rightarrow E \in \mathbb{R}^n$, where φ(P) must be continuous; alongside the usual weaker versions of continuity.

Remark 4

The nonvanishing $v(x) \neq 0$ condition will imply Farkas' lemma for the fixed-point existence for the economic case with nonnegative-price feasibility condition.

Economic Proof via Brouwer's Fixed-Point Theorem(Munkres, 2016): If $f : B^n \rightarrow B^n$ is a continuous function then there exists a point $x \in B^n$ such that f(x) = x.

Proof2

For obtaining a contradiction, let's assume $f(x) \neq x$ for every $x \in B^n$. Assume that x is a technology t+ on the frontier of the economy while the economy is taken as a unit ball $B^n \subset \mathbb{R}^n$. And f(x) is the market development m+ of t+. There must be a point in B^n where f(x) = x. Let's define vector field v(x) as a nonvanishing vector field which is p+ price system map or a profitability map which corresponds to p+ = (m+ - t+) as v(x) = (f(x) - x), while p+ or v(x) being a nonvanishing vector field $p+ \neq 0$, the non-zero condition, in B^n. There must be a point that violates the non-zero condition and that is S^1 boundary at the unit circle or unit ball with an x on the boundary where the vector field v(x) must point directly outward with no corresponding f(x) or with a t+ with no corresponding m+. Let's say v(x) is p+ the nonvanishing vector field as price maps of new technology t+ with market development m+. Take v(x) = ax where a > 0 and v(x) = (f(x) - x) = ax and then f(x) = (1 + a)x, the additive scaling at the boundary, but it violates v(x)'s definition as (f(x) - x) or violates market development m+ always staying greater than the t+ for the non-zero condition. So f(x) = (1 + a)x implies that $f(x) \notin B^n$ anymore: which is

[1] *The strict inequality (n, b) < 0 means that a pricing n for such non-commodity b is not only zero but negative. This has a subtle implication that when b is not in C then it means it lies in C∗ which implies a mispricing such that it is either a negative consumption or a negative price. This if compromises the convexity of C∗ it also does that for C. For further perspective see Debreu in (Debreu, 1959). Still the simple and general implication is sufficient here which is that: a nonnegative price must correspond to a nonnegative commodity. This implies general convexity and completeness of commodity space with necessarily continuous price correspondences (Balasko, 2009) as in Walras' law.*

a contradiction. So f(x) = x for some $x \in B^n$ and has a fixed-point. (Munkres, 2016)
Brouwer fixed point theorem: Let $f : K \rightarrow K$ be a continuous function from compact convex set K to itself. Then f has a fixed point. (Bon-Soon, 2020)

Lemma 3
Let D be a nonempty closed technology subset of a convex metric
space X as a market development level space, if it is apt to consider economy as a convex metric space. The inward set at x is defined as $ID(x) = \{w \in X : w = x \text{ or } y = W(w, x, 1/\pi) \text{ for some technology } y \in D \text{ and } \pi \geq 1\}$. Where $w \in X$ is a pricing corresponding to y.
An element $x \in X$ is called the fixed-point of the multivalued mapping $T : X \rightarrow CB(X)$ if $x \in T(x)$. A multivalued mapping T is said to be weakly inward on D if $Tx \subseteq$ the closure of ID(x) for $x \in D$ while T is here taken as a functional on market development level mapping it up to D. An element $y \in D$ is called an 'element of best approximation' of $x \in X$ (by the elements of the set D) if we have d(x,y) = d(x, D). (Beg & Abbas, 2006)

Proposition 2
The d(x, y) = d(x, D), above in Lemma3, can be conceived of as a convex differential as a Euclidean Distance D = || d+ - d ||, for some d(p, q) and d+(p+, q+), with possible fixed-points between subsets of an economy.

Sperner's Lemma
Given a homeomorphism assumption between a unit disk and a polytope simplex as a triangle, in a Sperner-labeled triangle, Δ, further triangulated into smaller triangles, $\Delta \in \Delta$, there exists an 'odd number', $i \neq 2, 4, 6,..n$, of smaller triangles, Δi, with the same vertices labeling as of the original triangle, Δ.

This is like a micro patch model inside the original model polytope of the triangle here. This implies a fixed-point such that this micro patch of the original model is the equilibrium point of the structure. (Schwartz, 2016)

Remark 5
The triangle polytope is an interesting case for economic application. Consider from Sperner's lemma:
"(Gale-Nikaido-Debreu GND lemma: strong version) Let Δ be the unit-simplex
of $\mathbb{R}^N$. Let ζ be an upper semicontinuous correspondence with non-empty, compact,
convex values from Δ into $\mathbb{R}^N$. Suppose ζ satisfies the following condition:
$\forall p \in \Delta, \forall z \in \zeta(p), p \cdot z \leq 0$.
Then there exists a $p \in \Delta$ such that $\zeta(p) \cap \mathbb{R}^N_- \neq \emptyset$".
And then again as a restatement of GND: "(Gale-Nikaido-Debreu) Let S denote the unit-sphere, for the norm $\|\cdot\|_2$ of $\mathbb{R}^N$. Let ζ be an upper semicontinuous correspondence from $S \cap \mathbb{R}^N_+$ in $\mathbb{R}^N$ which satisfies
$\forall\ q \in S \cap \mathbb{R}^{N+}$,
$\forall\ z \in \zeta(q), q \cdot z \leq 0$.
Then,
$\exists\ q \in S \cap \mathbb{R}^N_+$,
such that $\zeta(q) \cap \mathbb{R}^N_- \neq \emptyset$". (Le et al, 2020)
This latter "$q \cdot z \leq 0$" implies the nonvanishing vector field 'economic proof' in Proof2 above. Similarly it relates with economic proof in Proof3 below.

Proof 3
Sperner's via No-Retraction (Schwartz, 2016)(Harper, 2009): Given a $\Delta \in \mathbb{R}^2$ with its boundary as dΔ, a continuous function as retraction $r : \Delta \rightarrow d\Delta$ is not possible.
Proof: There is an f which maps vertices of each small triangle, in the big triangle which is labeled on vertices as {1, 2, 3}. Suppose this f is a retraction, for contradiction.
$|f(a) - f(b)| < 1\ \forall\ a, b \in \Delta$
and $|a - b| < p$ where p is such that every small triangle, in triangulation of the big triangle, has length < p. The f maps each small triangle so that

they are each within 1 of each other. (To be continued)

Remark 6

It corresponds with Remark5 and Proof2. Take the f to be a continuous price map in the commodity space a, b…, and then allow f to retract to the boundary of the convex simplectic market, the f must get negative (which here means the violation of Sperner labeling) for every a or b in Δ: f cannot be negative as price cannot because it violates the free disposal assumption in Economic theory.

(Proof continues):

But the side length < p for each small Δ. So no small triangle can have a Sperner labeling with vertices as {1, 2, 3} which gives us the contradiction.

Nonnegative Price Theorem

Given a free disposal assumption, a technology T+ on the boundary of an economy E and a continuous price correspondence P ∈ E,

P(T+) = 0.

Proof 4

"Farkas Lemma: If C is a closed convex cone, then for any $b \in \mathbb{R}^n \setminus C$ there is $n \in C*$ such that $(n, b) < 0$. Where $C*$ is the polar dual of C." (McLennan, 2018)

Proposition 3

Zorn's lemma through Well-ordering theorem, due to the 'nonnegative price' economic implication, implies Farkas' lemma at least for the case of assumption of a feasibly convex economy.

Well-Ordering Theorem (in place of Zorn's Lemma)

Zermelo: For every set X there is a well-ordering on X.

Proof: Given a partial order on X if every subset chain x ∈ X has a least element then X is well-ordered.

Remark 7

Farkas lemma implied through a nonnegative price condition provides the 'nonnegative price as the least-element condition' for economy (because price space is the dual of the commodity space).

Schauder's fixed-point theorem[2] (Schechter, 1997) extends the Brouwer's fixed-point theorem into Banach spaces while Zorn's lemma (also implying Well-ordering Theorem and Axiom of Choice) is traditionally used to prove Hahn-Banach Theorem in Banach spaces. Given that, the limit of a convergent subsequence, which "depends upon axiom of choice", is a fixed-point because of continuity. (McLennan, 2018)

Discussion & Analysis

The methodology used here has the following logical plot. If price functions cannot be negative, with convexity and continuity assumptions among different economies, then for any new technologies with developed returns should have a developed market level for their products. This market development can be more plausibly *local than global*. Suppose an importing economy i increases its technological development and develops new markets. This technological development is only possible when there is at least a local consumption market already developed for the product x of this new technology. Only this way this new technology can have returns. This in turn is only possible if this new technology is already linked to the product x that the economy i imports. Now, the economy j that

[2] *"Schauder's Fixed Point Theorem. Any continuous self-mapping of a compact convex subset of a Banach space has at least one fixed point."*

exports those products, to the economy i, must have its own locally developed production and consumption markets for the product x too. And this links economies i and j with n-economies through continuous functions as *convex differentials*. If the economy i needs the imports x from the economy j for developing a new technology to produce x at home then the imports of i from j are necessary for i to develop a new technology. Thus in order for an importing economy i to have its exports increased it needs precisely those imports of x from the exporting economy j. The reason for this, as given in Nonnegative Price Theorem below, is that a truly new technology, let's call it t+, must remain *unpriced* [3] as p(t+) = 0, because t+ technology's returns are undeveloped. And the reason for this is that the t+ does not have a developed consumption market for its potential products. Hence, exports and imports are complements, not substitutes. This means for any *priced* technology t in economy i that has developed returns it must have its consumption market for the product x in economy i while x is being imported from the economy j. So t in i depends on economy j for technical learning while i provides it with a domestic market for the potential product x that is similar to the x imported from j.

The question arises, if imports from j must increase for i's exports to increase then the net-importing status of economy i might never change. The first answer to this is that this paper considers an n-economies case and if labor costs in economy i are lower than those in j, k,...n-economies then there can be a possibility that i exports homemade x to economies k, l, m, or it can even out-compete the economy j in producing x that i used to import from j. The second answer is that the living standards and labor cost can be realistically assumed to monotonically increase in a relatively technologically advanced economy like j (for instance, the US) as compared to i (like China). The exports competitiveness of any given economy must decrease with its rising living standards and increasing labor costs as compared to the economies with lower living standards and labor costs. The labor costs in China have now started to increase too after the labor costs' increase in the US earlier. Thus i becomes a net-exporting economy when economies like j are facing steeper labor costs.

CONCLUSION

The Results above, relying on the proposition of Nonnegative Price Theorem, imply that marginally the new technological developments, which are tied to imports for an internationally less competitive economy, increase any given economy's exports' competitiveness because as per the Nonnegative Price Theorem a *truly new technology* is bound to stay *unpriced*; which is the implication of the convex differentials theorized here.

Disclosure Statement

Everything related to this article has been disclosed in the submission.

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[3] As prices cannot be negative for continuity and convexity assumptions, as amply proven below, therefore commodities and technologies also cannot be negative due to the free disposal assumption. But if a truly new technology has no developed returns it implies that an expenditure on this technology becomes a *consumption* instead of production. Yet such consumption is a *negative* consumption which means it is just a *waste* or a *bad* instead of a good. A *bad* implies *negative prices* as a violation of free disposal. Now, because prices cannot be negative, the truly new technology with no developed returns remains unpriced.

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Gomez-Sanchez et al (2021) Received: 14 September 2021 | Accepted: 20 September 2021 DOI: 10.1111/roie.12582
SPECIAL ISSUE PAPER Are importing and exporting complements or
substitutes in an emerging economy? The case of Colombia
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