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SIZED-BIASED SAULEH DISTRIBUTION AND APPLICATIONS ON LIFETIME DATA SETS

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ABSTRACT
The weighted distributions are generally utilized in numerous fields like medication, nature, and reliability, and for the improvement of legitimate measurable models. Weighted distributions have great importance towards efficient statistical models. In this research paper we used Sauleh distribution and introduced a new weighted distribution named sized biased Sauleh distribution (SB-SD) and derived many of its properties like mean, variance, moment generating function, moments, reliability analysis, Lorenz, and Bonferroni curve, mean deviation, and order statistic for the newly derived model. To estimate the parameters of the SB-SD maximum likelihood estimations methos is used. Finally, we used the SB-SD in two lifetime data sets to check the applicability of the newly derived model. It is concluded that the SB-SD is more flexible as compared to the Sauleh and some other well-known distributions. Therefore, the proposed model can be used to estimate the probabilities that a patient will survive until a time, mean lifetime, mean residual lifetime, instantaneous failure rate and further survival measures for the lifetime data.

Key Words: weighted, size-biased, Bonferroni and Lorenz curve, SB-SD, reliability

INTRODUCTION
The concept of weighted distributions gives us an approach to the issues related to the specification of the model and understanding of the data. When sample can be selected from the developed and the original distribution it gives a strategy to fitting models which have functions of unknown weight. In a situation where a data has differing probabilities for taking part in the sample, a population under consideration does not provide a normal random sample. Or give us the bias sample. Weighted distributions consider the strategy for ascertainment bias, by changing the probabilities of the actual event of occurrence to show up at those probabilities of specification whose event are being recorded and observed. The weighted distributions happen often in the investigations associated with unwavering quality and for the improvement of legitimate models of statistical.

The concept and idea of weighted distribution was firstly introduced by Fisher (1934) and later formulated by Rao (1965). Fisher concentrated how the strategies for ascertainment can impact the type of the recorded observation distribution. and afterward Rao presented and defined it overall terms regarding statistical modeling information when the typical act of utilizing standard distributions were discovered to be inadmissible. Weighted distributions are used to regulate the observed events of probabilities.

The weighted distributions are obtained by

$$f_w(x) = \frac{w(x)f(x)}{E\left(w(X)\right)} \tag{1.1}$$

Length-biased or size-biased distributions is a type of weighted distribution. Size-biased distributions are a particular occurrence of the wider set up which is known as weighted distributions. Basically, size-biased are extremely helpful when the sampling observation are taken relative to the size plans with inconsistent likelihood. The idea of sampling size-biased was presented by Cox (1969) and Zelen (1974) and that idea was that the distribution of size-biased can be utilized in evaluation of actual models for long run information. When the function of weight only observes the size/length of the units. Scheaffer (1972) generalized size-biased sampling which occurs when sampling multi-sized unites.

If we take, $w(x) = x$, then (1.1) becomes as

$$f_w(x) = \frac{xf(x)}{E(X)} \quad (1.2)$$

Where $E(X) = \mu_1'$, is the mean of the random variable X.

Ayesha (2017) discussed size-biased Lindley distribution. Khan et al. (2019) had researched on weighted modified Weibull distribution which is a new characterized parameter of three lifetime model. Al-Omari et al (2018) in their current paper offers another addition to the Ishita distribution called Size Biased Ishita distribution (SBID) It is shown that the SBID is the most suitable model for this informational set when contrasted with other distribution. Ganaie and Rajagopalan (2020) in their investigation have introduced a weighted three parameters new generalization of two parameters Akash distribution. And compared three parameters with two parameter Akash distribution and by using criterion values infer that the weighted three parameters Akash distribution prompts a preferable fit than the two parameters Akash. Shukla and Shanker (2020) has proposed lifetime modelling data a weighted Pranav distribution which includes one parameter Pranav distribution as an exceptional case. The fitted plot of distributions showed that WPD is a better model among the one-parameter exponential, Lindley and Pranav distributions and two-parameter Weibull, EP and WLD. Santoro (2022) focused on the model generated by the skew-normal distribution, called Extended Half Skew-Normal distribution. Shama (2023) proposed Modified generalized Weibull distribution: theory and applications.

Aijaz et al (2020) introduced one parameter Sauleh distribution which is a mixture of exponential distribution having scale parameter and gamma distribution having scale parameter with fixed shape parameter as 4. The probability distribution function (pdf) of Sauleh distribution is given by,

$$f(x;\theta) = \frac{\theta^4}{\theta^4 + 2\theta + 6}(\theta + x^2 + x^3)e^{-\theta x}; \; x > 0, \theta > 0 \quad (1.3)$$

The mean of the Sauleh distribution is

$$Mean = \mu'_1 = \frac{(\theta^4+6\theta+24)}{\theta(\theta^4+2\theta+6)} \quad (1.4)$$

Formulation of Model "Size-Biased Sauleh Distribution (Sb-Sd)" And Some Properties

Using equations (1.3) & (1.4) in equation (1.2) we get the size-biased Sauleh distribution (SB-SD) having one shape parameter with the following pdf

$$f(x;\theta) = \frac{\theta^5}{\theta^4 + 6\theta + 24} x(\theta + x^2 + x^3)e^{-\theta x}; \quad x > 0, \theta > 0 \quad (2.5)$$

The cumulative distribution function (cdf) of the SB-SD is given by,

$$F(x) = 1 - \frac{(e^{-\theta x}(\theta^4x^4+\theta^4x^3+\theta^5x+4\theta^3x^3+3\theta^3x^2+\theta^4+12\theta^2x^2+6\theta^2+24\theta x+6\theta+24)}{\theta^4+6\theta+24} \quad (2.6)$$

Figure 2.1

pdf of SB-SD under different values of θ

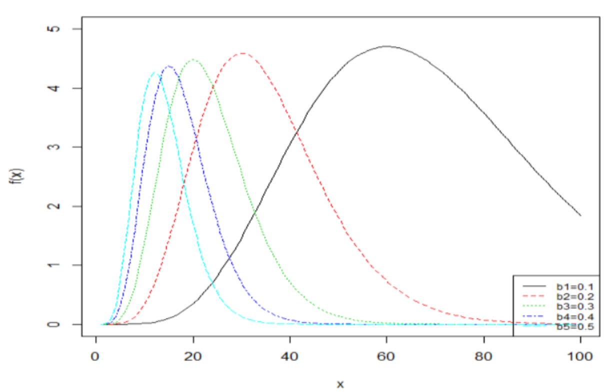
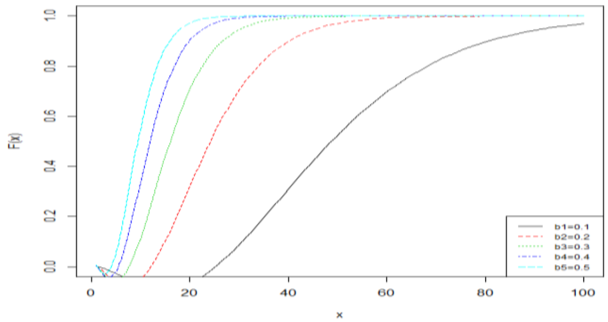


Figure 2.2
cdf of SB-SD under different values of θ



The moment generating function (mgf) of the SB-SD is

$$E(e^{tx}) = \frac{\theta^4\left(1-\frac{t}{\theta}\right)^3+6\theta\left(1-\frac{t}{\theta}\right)+24}{(\theta^4+6\theta+24)\left(1-\frac{t}{\theta}\right)^5} \qquad (2.7)$$

The r^{th} moment about the origin of a SB-SD is

$$E(X^r) = \mu_r'$$

$$= \frac{1}{\theta^4+6\theta+24}\left(\frac{\theta^4\Gamma(r+2)+\theta\Gamma(r+4)+\Gamma(r+5)}{\theta^r}\right) \qquad (2.8)$$

Substituting $r = 1, 2, 3$ & 4, we will get the first four raw moments of the SB-SD as

$$Mean = \mu_1' = \frac{2(\theta^4+12\theta+60)}{\theta(\theta^4+6\theta+24)}$$

$$\mu_2' = \frac{6(\theta^4+20\theta+120)}{\theta^2(\theta^4+6\theta+24)}$$

$$\mu_3' = \frac{24(\theta^4+30\theta+210)}{\theta^3(\theta^4+6\theta+24)}$$

$$\mu_4' = \frac{120(\theta^4+210\theta+336)}{\theta^4(\theta^4+6\theta+24)}$$

The variance and standard deviation (SD) of the SB-SD is given as

$$Var(X) = \frac{2(\theta^8+30\theta^5+192\theta^4+72\theta^2+720\theta+1440)}{\theta^2(\theta^4+6\theta+24)^2} \qquad (2.9)$$

Then the standard deviation is given by,

$$SD = \sigma = \frac{\sqrt{2(\theta^8+30\theta^5+192\theta^4+72\theta^2+720\theta+1440)}}{\theta(\theta^4+6\theta+24)} \qquad (2.10)$$

Similarly, the third moment (μ_3) and fourth moments (μ_4) about mean are given as

$$\mu_3 = \frac{4\theta^{12}-3672\theta^9+1728\theta^8-100224\theta^6-550368\theta^5+138240\theta^4-464832\theta^3-4639680\theta^2-11093760\theta+138240}{\theta^3(\theta^4+6\theta+24)^3} \qquad (2.11)$$

$$\mu_4 = \frac{120(\theta^4+210\theta+336)}{\theta^4(\theta^4+6\theta+24)} - 4\left(\frac{24(\theta^4+30\theta+210)}{\theta^3(\theta^4+6\theta+24)}\right)\left(\frac{2(\theta^4+12\theta+60)}{\theta(\theta^4+6\theta+24)}\right) + 6\left(\frac{6(\theta^4+20\theta+120)}{\theta^2(\theta^4+6\theta+24)}\right)^2\left(\frac{2(\theta^4+12\theta+60)}{\theta(\theta^4+6\theta+24)}\right) - 3\left(\frac{2(\theta^4+12\theta+60)}{\theta(\theta^4+6\theta+24)}\right)^4 \qquad (2.12)$$

The co-efficient of variation (CV) of the SB-SD is

$$C.V = \frac{\sigma}{\mu_1'}$$

$$C.V = \frac{\sqrt{2(\theta^8+30\theta^5+192\theta^4+72\theta^2+720\theta+1440)}}{2(\theta^4+12\theta+60)} \qquad (2.13)$$

And,

$$\gamma = \frac{\sigma^2}{\mu_1'}$$

$$\gamma = \frac{(\theta^8+30\theta^5+192\theta^4+72\theta^2+720\theta+1440)}{\theta(\theta^4+6\theta+24)(\theta^4+12\theta+60)} \qquad (2.14)$$

The skewness and kurtosis of the SB-SD are given as

$$\beta_1 = \frac{\left(4\theta^{12}-3672\theta^9+1728\theta^8-100224\theta^6-550368\theta^5+138240\theta^4-464832\theta^3-4639680\theta^2-11093760\theta+138240\right)^2}{4\theta(\theta^4+6\theta+24)(\theta^8+30\theta^5+192\theta^4+72\theta^2+720\theta+1440)^2} \qquad (2.15)$$

$$\beta_2 = \frac{\frac{120(\theta^4+210\theta+336)}{\theta^4(\theta^4+6\theta+24)}-4\left(\frac{24(\theta^4+30\theta+210)}{\theta^3(\theta^4+6\theta+24)}\right)\left(\frac{2(\theta^4+12\theta+60)}{\theta(\theta^4+6\theta+24)}\right)+6\left(\frac{6(\theta^4+20\theta+120)}{\theta^2(\theta^4+6\theta+24)}\right)^2\left(\frac{2(\theta^4+12\theta+60)}{\theta(\theta^4+6\theta+24)}\right)-3\left(\frac{2(\theta^4+12\theta+60)}{\theta(\theta^4+6\theta+24)}\right)^4}{\left(\frac{2(\theta^8+30\theta^5+192\theta^4+72\theta^2+720\theta+1440}{\theta^2(\theta^4+6\theta+24)}\right)^2} \qquad (2.16)$$

The mean deviation about mean of the SB-SD is

$$D(\mu) = 2\mu\, F(\mu) - 2\int_0^{\mu} x\, f(x)dx$$

$$D(\mu) = \frac{2e^{-\theta\mu}\left(\theta^5(\mu+\mu^2-\mu^3+\mu^4-\mu^5)+\theta^4(2-\mu^2+4^3+\mu^4)+\theta^3(8\mu^3+6\mu^2)+\theta^2(36\mu^2+18\mu)+\theta(96\mu+24)+120\right)}{\theta(\theta^4+6\theta+24)} \qquad (2.17)$$

The mean deviation about median of the SB-SD is

$$D(M) = \mu - 2\int_0^{M} x\, f(x)dx$$

$$D(M) = \frac{2e^{-\theta M}(M^2\theta^6+2M\theta^5+2\theta^4+\theta^5M^4+4\theta^4M^3+12\theta^3M+24\theta^2M+24\theta+\theta^5M^5+5\theta^4M^4+20\theta^3M^3+60\theta^2M^2+120\theta M+120)}{\theta(\theta^4+6\theta+24)} - \mu$$

(2.18)

The concept of Bonferroni is used in the life testing or reliability test this curve differentiate the mean of the total mean with lower group. The Bonferroni curve of the SB-SD is

$$B(s) = \frac{1}{s\mu}\int_0^t x\, f(x)\, dx$$

$$B(s) = \frac{1}{s\mu}\int_0^t x\, \frac{\theta^5}{\theta^4 + 6\theta + 24} x(\theta + x^2 + x^3)e^{-\theta x}\, dx$$

$$B(s) = \frac{1}{s}\left(1 - \frac{e^{-\theta t}(\theta^6 t^2 + \theta^5(2t + t^4 + t^5) + \theta^4(2 + 4t^3 + 5t^4) + \theta^3(12t^2 + 2\theta t^3) + \theta^2(24t + 60t^2) + 144\theta + 120)}{\theta(\theta^4 + 6\theta + 24)\mu}\right) \quad (2.19)$$

Lorenz curve indicates the distribution, or the process inequalities are generally used to quantify these inequalities across a population. If Lorenz curve drifts away from the baseline the distribution unequal level got increases. The Lorenz curve of the SB-SD is

$$L(s) = \frac{1}{\mu}\int_0^t x\, f(x)\, dx$$

$$L(s) = \frac{1}{\mu}\int_0^t x\, \frac{\theta^5}{\theta^4 + 6\theta + 24} x(\theta + x^2 + x^3)e^{-\theta x}\, dx$$

$$L(s) = \left(1 - \frac{e^{-\theta t}(\theta^6 t^2 + \theta^5(2t + t^4 + t^5) + \theta^4(2 + 4t^3 + 5t^4) + \theta^3(12t^2 + 2\theta t^3) + \theta^2(24t + 60t^2) + 144\theta + 120)}{\theta(\theta^4 + 6\theta + 24)\mu}\right)$$

(2.20)

Reliability Measures and Order Statistics

In this section we are presenting few reliability measures like reliability/survival function, hazard function, The reliability function or the survival function of the SB-SD is

$$R(x) = \frac{e^{-\theta x}(\theta^4 x^4 + \theta^4 x^3 + \theta^5 x + 4\theta^3 x^3 + 3\theta^3 x^2 + \theta^4 + 12\theta^2 x^2 + 6\theta^2 + 24\theta x + 6\theta + 24)}{\theta^4 + 6\theta + 24} \quad (3.1)$$

The hazard function or failure rate of the SB-SD is

$$h(x) = \frac{\theta^5 x(\theta + x^2 + x^3)}{(\theta^4 x^4 + \theta^4 x^3 + \theta^5 x + 4\theta^3 x^3 + 3\theta^3 x^2 + \theta^4 + 12\theta^2 x^2 + 6\theta^2 + 24\theta x + 6\theta + 24)}$$

Figure 3.1

graph of reliability function of SB-SD

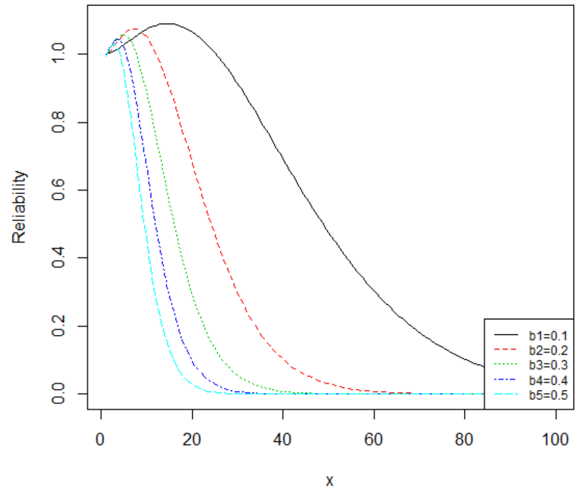
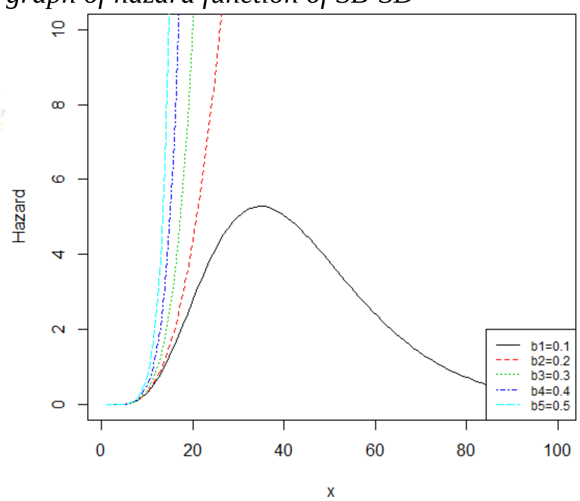


Figure 3.2

graph of hazard function of SB-SD



From figure 1.4, it can be seen that the hazard function for the SB-SD is increasing monotonically only for $\theta = 0.1$ it has increasing and decreasing trend.

Order statistics are test esteems put in ascending order. The investigation of order statistics manages the uses of these arranged qualities and their functions. Pdf of minimum order statistics form SB-SD is

$$f_{x_1} = \frac{n\theta^5(\theta x+x^3+x^4)e^{-\theta x}}{n\theta^5(\theta x+x^3+x^4)e^{-\theta x}}\left(1-\left(\frac{e^{-\theta x}(\theta^5+\theta^4(x^4+x^3+1)+\theta^3(4x^3+x^2)+\theta^2(12x^2+6x)+\theta(24x+6)+24)}{\theta^4+6\theta+24}\right)^{n-1}\right) \quad (3.3)$$

Estimation by Maximum Likelihood Estimation (Mle)

For the estimation of parameter of SB-SD we used maximum likelihood estimation. The likelihood function for the SB-SD is

$$L = \prod_{i=1}^{n}\left(\frac{\theta^5}{\theta^4+6\theta+24}\right)x(\theta+x^2+x^3)e^{-\theta x} \quad (4.1)$$

Taking log likelihood of (4.1) and differentiating it w.r.to θ, we get

$$\frac{\delta \ln L}{\delta\theta} = \frac{5n}{\theta} - \frac{n(6\theta^3+6)}{\theta^4+6\theta+24} + \sum_{i=1}^{n}\left(\frac{1}{\theta+x_i^2+x_i^3}\right) - n\bar{x} \quad (4.2)$$

The maximum likelihood estimates (MLEs) $\hat{\theta}$ of θ is,

$$\frac{5n}{\theta} - \frac{n(6\theta^3+6)}{\theta^4+6\theta+24} + \sum_{i=1}^{n}\left(\frac{1}{\theta+x_i^2+x_i^3}\right) - n\bar{x} = 0 \quad (4.3)$$

Where $\bar{x}$ is the sample mean.

APPLICATIONS

In this section, we give an application of the size biased Sauleh distribution to real informational collection. We utilize the past informational collection to get the estimates of ML for each model parameters and afterward relating the outcomes with statistic of goodness-of-fit AIC (Akaike information criterion), AICC (Akaike information criterion corrected), and BIC (Bayesian information criterion). The altered model relates to more modest AIC, AICC, and BIC esteems. The model related to the fit of the new model and have subsidiary AIC, BIC and AICC esteems is better, Weighted Sauleh distribution with the Sauleh, Pranav, Akash, Ishita and Lindley distribution.

Dataset 1

The data set represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed, and reported by Bjerkedal (1960). The data are as follows:

0.1, 0.33, 0.44, 0.56, 0.59, 0.59, 0.72, 0.74, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55, 2.54, 0.77.

Table 1

Goodness of fit tests for the survival times (in days) of 72 guinea pigs

Distributions	Estimates	-2logL	AIC	AICC	BIC
Size-Biased Sauleh Distribution	**2.1096 (0.0958)**	**194.7482**	**196.7482**	**196.8053**	**196.6055**
Sauleh Distribution	1.4976 (0.0738)	214.7476	216.7476	216.8047	216.6049
Akash Distribution	1.2227 (0.0815)	216.1373	218.1373	218.1947	217.9946
Ishita Distribution	0.8847 (0.0592)	219.2440	221.2440	221.3011	221.1013
Pranav Distribution	1.4780 (0.0689)	225.0703	227.0703	227.1274	226.9276
Lindley Distribution	1.8744 (0.0771)	227.0510	229.0510	229.1081	228.9083

Dataset 2

This data set represents the lifetime data relating to relief times (in minutes) of 20 patients receiving an analgesic and reported by Gross and Clark. The data are as follows:

1.1,1.4,1.3,1.7,1.9,1.8,1.6,2.2,1.7,2.7,4.1,1.8,1.5,1.2,1.4,3.0,1.7,2.3,1.6,2.0

Table 2

Goodness of fit tests for the relief times (in minutes) of 20 patients

Distributions	Estimates	-2logL	AIC	AICC	BIC
Size-Biased Sauleh Distribution	**1.9834 (0.1726)**	**49.4232**	**51.4232**	**51.6454**	**50.7242**
Sauleh Distribution	1.4151 (0.1335)	58.1930	60.1930	60.4152	59.4140
Akash Distribution	1.1569 (0.1455)	59.5226	61.5226	61.7448	60.8236
Ishita Distribution	0.8233 (0.1074)	58.3940	60.3940	60.6162	59.6950
Pranav Distribution	1.4014 (0.1247)	62.3860	64.3860	64.6082	63.6870
Lindley Distribution	0.8161 (0.1360)	60.5000	62.5000	62.7222	60.8010

Table 3

Summary Statistics for SB-SD for data 1 & 2

Measures	For data set 1	For data set 2
	$\theta = 2.1096$	$\theta = 1.9834$
Mean	3.7235	3.8647
Variance	30.1805	35.6026
Standard Deviation	5.4937	5.9668
Co-efficient of Variation	1.4754	1.5439
Skewness	1.0422	1.0881
Kurtosis	107.6072	129.9532

Figure 1.5

fitted pdf for the survival times

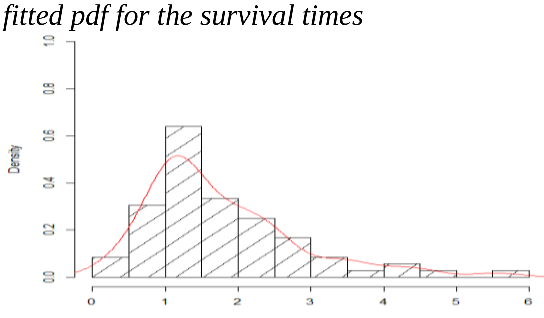
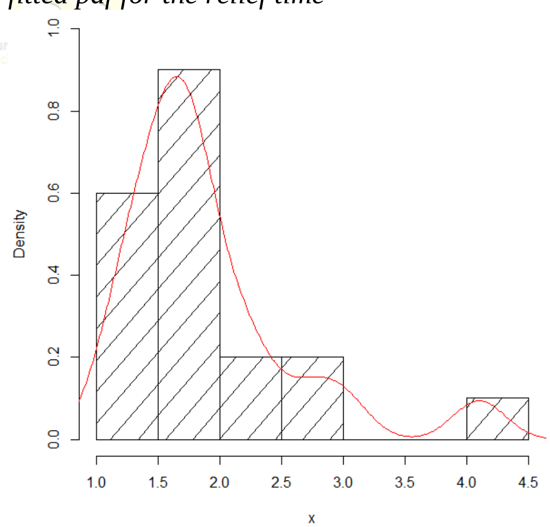


Figure 1.6

fitted pdf for the relief time



CONCLUSION

In this article we introduced a weighted distribution named as size-biased Sauleh distribution (SB-SD). We have derived various properties like mean, variance, moment generating function, moments, reliability measures (reliability function and hazard function), Lorenz, and Bonferroni curve, mean deviation (from mean and median) and order statistic. SB-SD has a single parameter and from different values of the parameter the SB-SD is positively skewed. The hazard function for the SB-SD is increasing monotonically for different values of θ but only for $\theta = 0.1$ it has increasing and decreasing trend.

Picking a model with a parameter can make unrealistic assumption which sometimes leads to high bias and poor prediction. Such models are not adaptable enough to depict the population well. So, to fit the observe information a criterion is used like Akaike's Information Criterion (AIC) and the Bayesian Information Criterion (BIC), are broadly utilized for model determination. We compared SB-SD with Sauleh, Akash, Ishita, Pranav and Lindley distribution. By looking at the result based in table 1 and 2 we concluded that SB-SD has smaller AIC, AICC and BIC as compared to Sauleh, Akash, Ishita, Pranav and Lindley distribution and therefore SB-SD conduct an improved fits for modeling lifetime datasets.

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